The 4th-order Runge-Kutta method

1. Use four steps of the 4th-order Runge-Kutta method to approximate a solution on the interval [0, 1] to the initial-value problem defined by

$$y^{(1)}(t) = -y(t) + t - 2$$
$$y(0) = 1$$

Answer: 1.0, 0.365234375, –0.07382869720, –0.3604769395, –0.52842320234

2. Use eight steps of the 4th-order Runge-Kutta method to approximate a solution on the interval [0, 1] to the initial-value problem defined that shown in Question 1.

Answer: To ten digits of significance, 1, 0.6549886068, 0.3652048910, 0.1241594432, –0.0738746219, –0.2339512641, –0.3605305889, –0.4575486265, –0.5284789123.

3. If the actual solution is $y(t) = 4e^{-t} + t - 3$, argue that this method is indeed $O(h^5)$ for a single step.

Answer: To four significant digits, the error of the approximation of $y(0.25)$ in Question 1 is 0.00003124 and the error of the approximation of $y(0.125)$ in Question 2 is 0.0000009964, and this second value is approximately one 32nd the error of the first.

4. If the actual solution is $y(t) = 4e^{-t} + t - 3$, argue that this method is indeed $O(h^4)$ over multiple steps.

Answer: $y(1) = 4e^{-1} + 1 - 3 \approx -0.5284822353142307136$, so the error of the approximation in Question 1 is approximately 0.00005903 while the error with the second approximation is 0.000003323, which is approximately one sixteenth that of the previous approximation.